A Fast, Linearized Jordan-Wigner Transform via Twisted Group Algebras

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Abstract

We present a unified algebraic framework relating the Heisenberg-Weyl algebra (qubits) and the Clifford algebra (fermions) through the lens of Twisted Group Algebras over $\mathbb{F}_2^{2n}$. We derive a compact, purely linear formulation of the Jordan-Wigner transform over the binary field. By introducing a specific integration operator $\mathcal{E}$, we provide explicit conversion formulas between Pauli codes and Clifford codes that include the exact sign phase via a cohomological scalar product. This formulation reduces the computational cost of the mapping from exponential matrix operations to $O(n)$ bitwise operations, making it suitable for high-performance quantum error correction simulations.

1 Introduction

In quantum information theory and mathematical physics, the descriptions of finite-dimensional quantum systems often rely on three isomorphic structures:

1. The **Clifford Algebra** $Cl(n, n)$, describing $2n$ Majorana fermions.
2. The **Heisenberg-Weyl Algebra**, describing n qubits (Pauli group).
3. The Algebra of Matrices $M_{2^n}(\mathbb{R})$ or $M_{2^n}(\mathbb{C})$.

While the isomorphism between these structures is guaranteed by the Stone-von Neumann theorem and the classification of Central Simple Algebras, explicit translations are often burdened by complex tensor product notations. We propose a geometric unification based on the **Twisted Group Algebra** of the elementary abelian group $V = \mathbb{F}_2^{2n}$. Within this framework, the Jordan-Wigner transform appears as a change of basis relating two distinct polarizations (symplectic and orthogonal) of the underlying vector space.

2 Mathematical Framework

2.1 The Twisted Group Algebra

Let $V = \mathbb{F}_2^{2n}$ be a binary vector space. We define the algebra $\mathcal{A}$ spanned by basis elements $\{\mathbf{e}_x \mid x \in V\}$ subject to the multiplication rule:

$$\mathbf{e}_x \cdot \mathbf{e}_y = (-1)^{\beta(x,y)} \mathbf{e}_{x+y} \tag{1}$$

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where $\beta : V \times V \to \mathbb{F}_2$ is a 2-cocycle. The physics of the system is determined by the choice of basis for V:

- **Weyl Polarization (Qubits):** A basis that diagonalizes the symplectic form. Elements are indexed by $(u, v) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$ corresponding to $X^u Z^v$.
- **Clifford Polarization (Fermions):** A basis that diagonalizes the quadratic form. Elements are generators satisfying $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$.

2.2 The Integration Operator $\mathcal{E}$

To express the non-local mapping between these bases efficiently, we introduce the **Exclusive Integration Operator** (or Strict Lower Triangular Integration) $\mathcal{E} : \mathbb{F}_2^n \to \mathbb{F}_2^n$.

Definition 1 (Operator $\mathcal{E}$). *For a vector $x \in \mathbb{F}_2^n$, the vector $\mathcal{E}(x)$ is defined by the exclusive prefix sum:*

$$(\mathcal{E}(x))_k = \bigoplus_{j=0}^{k-1} x_j \tag{2}$$

In matrix form, $\mathcal{E}$ is the strict lower triangular matrix of ones.

3 The Linearized Transform

We denote a Pauli operator by the triplet (u, v, s) representing the operator $(-1)^s X^u Z^v$ (following the convention X before Z). We denote a Clifford operator by (P, N, s') where P (Positives) and N (Negatives) encode the indices of the active Majorana generators, and s' is the global sign.

Theorem 1 (Pauli to Clifford Linearization). *The mapping $\Phi : (u, v, s) \mapsto (P, N, s')$ is explicitly given by:*

$$N = v \oplus \mathcal{E}(u) \tag{3}$$

$$P = u \oplus N \tag{4}$$

$$s' = s \oplus \langle N, \mathcal{E}(P) \rangle \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product over $\mathbb{F}_2^n$.

3.1 Physical Interpretation

- **Topology (N, P):** Equations (3) and (4) describe the non-local "string" structure. The integration $\mathcal{E}(u)$ propagates the influence of the Shift operator (X) into the Phase sector, creating the characteristic Jordan-Wigner strings.
- **Cohomology (s'):** The term $\langle N, \mathcal{E}(P) \rangle$ in Eq. (5) represents the *reordering cost.* It counts the parity of transpositions required to sort the fermionic generators into the canonical order. This term is precisely the coboundary linking the symplectic cocycle to the orthogonal cocycle.

Theorem 2 (Inverse Transform: Clifford to Pauli). *The inverse mapping is symmetric (an involution of the structure):*

$$u = P \oplus N \tag{6}$$

$$v = N \oplus \mathcal{E}(u) \tag{7}$$

$$s = s' \oplus \langle N, \mathcal{E}(P) \rangle \tag{8}$$

4 Conclusion

This formulation demonstrates that the complexity of the Jordan-Wigner transform is purely topological and can be handled via linear algebra over the binary field $\mathbb{F}_2$. By identifying the sign correction with the scalar product $\langle N, \mathcal{E}(P)\rangle$, we provide a closed-form solution suitable for high-performance simulation of fermionic systems and quantum error correction codes without the overhead of matrix multiplication.